On the dominance of Mallows model averaging estimator over ordinary least squares estimator

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ABSTRACT

This note studies Mallows model averaging method for finite sample size situation. Sufficient conditions under which the model averaging estimator dominates the ordinary least squares estimator are provided with respect to mean squared error.

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1. Introduction

Over the past two decades, there has been a substantial amount of interest in model averaging. Within the Bayesian paradigm, model averaging has long been a popular approach; see, for example, Hoeting et al. (1999) for a comprehensive review. In recent years, within the frequentest paradigm, model averaging methods have been proposed, including weighting strategies using scores of information criteria (Buckland et al., 1997; Hjort and Claeskens, 2003; Claeskens et al., 2006; Zhang and Liang, 2011; Zhang et al., 2012), asymptotically optimal methods (Hansen, 2007; Liang et al., 2011; Hansen and Racine, 2012; Liu and Okui, 2013), model averaging marginal regression (Li et al., 2015), among others. Frequentist model averaging technique has also been utilized in many contexts such as constructing optimal instruments (Kuersteiner and Okui, 2010), autoregressive models (Hansen, 2010), mixed-effects models (Zhang et al., 2014), and quantile regression models (Lu and Su, 2015).

However, in most of the existing literature, large sample properties on model averaging methods are mainly focused on and model averaging methods were generally compared asymptotically or by simulation performance (see, for example Liang et al., 2011, Hansen, 2014 and Liu, 2015). As far as we know, there was no analytical finite sample study on the condition under which a model averaging estimator dominates the ordinary least squares (OLS) estimator with respect to mean squared error (MSE). The main contribution of this paper is to develop results on the exact dominance of a model average estimator over the OLS estimator. The results are exact in the sense that they are valid for any sample size, especially when the sample size is small.

The remainder of this paper is organized as follows. Section 2 introduces some estimators and basic theoretic results. Section 3 presents the MSE comparison between the model average estimators and the OLS estimator, and provides the sufficient conditions under which these model averaging estimators dominate the OLS estimator. Section 4 concludes the paper. Technical proofs are contained in Appendix.

2. Estimation

We are concerned with a linear regression model:

\[ y_i = x_i^T \beta + e_i, \quad e_i \sim \text{Normal}(0, \sigma^2), \ i = 1, \ldots, n \]  

(2.1)

where \( y_i \) is a scalar dependent variable, \( x_i (p \times 1) \) are independent variables, \( \beta (p \times 1) \) is a coefficient vector, \( e_i \) is an error term, and \( (y_i, x_i) \) for \( i = 1, \ldots, n \) are assumed to be independent. Following
has full column rank.

Suppose that we have \( M \) groups of regressors. We combine \( M \) nested sub-models of (2.2) candidate models, and the \( m \)th candidate model includes the first \( m \) groups of variables of \( X \), denoted by \( X_m \). Denote the group size of the \( m \)th group by \( k_m \). Let \( v_m = \sum_{j=1}^{k_m} j \), and thus \( v_m \) is the number of variables used in the \( m \)th candidate model and is also the number of columns of \( X_m \).

Let \( \Pi_m \) be a selection matrix so that \( \Pi_m = (I_{k_m}, 1_{k_m} \times (p-v_{k_m})) \) and thus \( X_m = \Pi_m X \). Define a \( p \times p \) matrix \( \mathcal{A}_m = \Pi_m (X_m^\top X_m)^{-1} \Pi_m^\top \). Under the \( m \)th candidate model, the restricted OLS estimator of \( \beta \) can be written as

\[
\hat{\beta}_{OLS} = \mathcal{A}_m \hat{\beta}_{OLS}.
\]

For the \( M \)th candidate model, \( \hat{\beta}_M = \hat{\beta}_{OLS} \). The model averaging estimator of \( \beta \) is

\[
\hat{\beta}(\mathbf{w}) = \frac{\sum_{m=1}^{M} w_m \hat{\beta}_m}{\sum_{m=1}^{M} w_m},
\]

where \( w_m \) is the weight corresponding to the \( m \)th candidate model and \( \mathbf{w} = (w_1, \ldots, w_M)^\top \), belonging to weight set \( \mathcal{W} = \left\{ \mathbf{w} \in [0, 1]^M : \sum_{m=1}^{M} w_m = 1 \right\} \).

Let \( \mathbf{v} = (v_1, \ldots, v_M)^\top \). Hansen (2007) proposed choosing weights by minimizing Mallows’ criterion

\[
C(\mathbf{w}) = \|X\hat{\beta}(\mathbf{w}) - \mathbf{y}\|^2 + 2\sigma^2 \mathbf{w}^\top \mathbf{v}.
\]

Let \( \hat{\mathbf{w}} = (\hat{w}_1, \ldots, \hat{w}_M)^\top = \arg \min_{\mathbf{w} \in \mathcal{W}} C(\mathbf{w}) \), so that the combined estimator \( \hat{\beta}(\hat{\mathbf{w}}) \) is the Mallows model averaging (MMA) estimator of \( \beta \).

3. MSE comparison

Let

\[
q(\hat{\beta}_{OLS}, \hat{\sigma}^2) = I({m} < {M}) \left[ 2\sigma^2 (v_{m} - v_{m-1}) - \|X\hat{\beta}_m - \hat{\beta}_{m-1}\|^2 \right] + \sigma^4 \sum_{j=1}^{M-1} \left( (n - p - 2)(n - p - 1)(v_{m-1} - v_{m-2}) - 4 \right) \left( v_{m-1} - v_{m-2} \right),
\]

where \( I(\cdot) \) denotes the indicator function as usual. For any estimator \( \hat{\beta} \) of \( \beta \), its MSE is defined by

\[
E \|X\hat{\beta} - \mathbf{y}\|^2 = \sigma^2 + \sigma^4 \sum_{m=1}^{M} \left( \|\hat{\beta}_{OLS} - \hat{\beta}_m\|^2 \right).
\]

Theorem 1. \( E \left\|X\hat{\beta}(\hat{\mathbf{w}}) - \mathbf{y}\right\|^2 = \sigma^2 + \sigma^4 \sum_{m=1}^{M} \left( \|\hat{\beta}_{OLS} - \hat{\beta}_m\|^2 \right). \)

See Appendix A.1 for the proof of Theorem 1. From Theorem 1, we have the following result.

Corollary 1. If \((n - p - 2)(n - p)^{-1} k_m \geq 4\) for all \( m \geq 2 \), then

\[
E \left\|X\hat{\beta}(\hat{\mathbf{w}}) - \mathbf{y}\right\|^2 < E \left\|X\hat{\beta}_{OLS} - \mathbf{y}\right\|^2,
\]

i.e., \( \hat{\beta}(\hat{\mathbf{w}}) \) dominates \( \hat{\beta}_{OLS} \).

See Appendix A.2 for the proof of Corollary 1. We note that the result in Corollary 1 provides the exact dominance condition for the MMA estimator over the OLS estimator in the MSE sense. This result is valid for any sample size \( n \). In a special case, when \( n \) tends to infinity, Corollary 1 provides a sufficient condition for the MMA estimator dominating (asymptotically) the OLS estimator, which is \( k_m \geq 4 \) for all \( m \geq 2 \) (also see Hansen, 2014).

Irrespective of sample size, our Corollary 1 indicates that there is a scale \((n - p - 2)(n - p)^{-1} < 1 \) associated with \( k_m \) and when \((n - p - 2)(n - p)^{-1} k_m \geq 4 \) for all \( m \geq 2 \), the MMA estimator dominates the OLS estimator.

4. Concluding remarks

Firstly we have developed new results on deriving the condition under which the MMA estimator dominates the OLS estimator in the exact MSE sense. This exact condition depends on the sample size and the number of regressors. Secondly we remark that our theory is also confined to the context of nested models. Extension of the current analysis to non-nested models will be very challenging. Lastly, the MSE comparison of the current paper is built under the normally distributed and homoscedastic error. Developing MSE comparison under other error cases is also an interesting topic for future research.

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Appendix

A.1. Proof of Theorem 1

First, similar to Hansen (2014), we define cumulative weights

\[
w_m^* = w_1 + \cdots + w_m
\]

and \( \mathbf{w}^* = (w_1^*, \ldots, w_M^*)^\top \). Then, \( w_m = w_m^* - w_m^{*-1} \) for \( m \geq 2 \), \( w_1 = w_1^* \), and \( \mathbf{w} \in \mathcal{W} \) is equivalent to

\[
\mathbf{w}^* \in \mathcal{W}^* \equiv \left\{ \mathbf{w}^* \in [0, 1]^M : 0 \leq w_1^* \leq \cdots \leq w_M^* = 1 \right\}
\]

and the model averaging estimator \( \hat{\beta}(\mathbf{w}) \) can be rewritten as

\[
\hat{\beta}(\mathbf{w}) = \hat{\beta}_{OLS} - \sum_{m=1}^{M-1} w_m^* (\hat{\beta}_{m+1} - \hat{\beta}_m).
\]

Let \( \hat{w}_m^* = \hat{w}_1 + \cdots + \hat{w}_m, \hat{\mathbf{w}}^* = (\hat{w}_1^*, \ldots, \hat{w}_M^*)^\top \).

\[
b_m = \left\|X\hat{\beta}_m\right\|^2.
\]

and

\[
C(\mathbf{w}^*) = \sum_{m=1}^{M-1} \left\{ w_m^2 (b_m - b_{m+1}) - 2\sigma^2 w_m^4 k_{m+1} \right\}.
\]
From Lemma 1 of Hansen (2014), we have
\[ C(\mathbf{w}) = C^*(\mathbf{w}^*) + b_m + 2\sigma^2 v_m \] (A.5)
and
\[ \mathbf{w}^* = \arg \min_{\mathbf{w}^* \in \mathcal{X}^*} C^*(\mathbf{w}^*). \] (A.6)

Hence, from (2.5) and (A.3)–(A.6), we know that both \( \hat{\mathbf{w}} \) and \( \mathbf{w}^* \) depend on \( y \) through \( \hat{\beta}_{\text{OLS}} \) and \( \hat{\sigma}^2 \).

Since weights \( \hat{w}_1, \ldots, \hat{w}_M \) are determined by data, the indexes of candidate models with positive weights are random. We use \( \{m_1(y), \ldots, m_M(y)\} \) to denote the indexes set, where \( f(y) \) and \( m_{ij}(y) \) depend on \( y \). By the analysis of the above paragraph, we know that \( C(\mathbf{w}) \) depends on \( y \) through \( \hat{\beta}_{\text{OLS}} \) and \( \hat{\sigma}^2 \), so we can write \( f(y) \) and \( m_{ij}(y) \) as \( f(\hat{\beta}_{\text{OLS}}, \hat{\sigma}^2) \) and \( m_{ij}(\hat{\beta}_{\text{OLS}}, \hat{\sigma}^2) \). Instead, for simplicity, we write them as \( f \) and \( m \), although they are random. Thus, \( w_{m_1} = \cdots = w_{m_{j-1}} \) and \( w_{m_j} = 1 \). From the proof of Theorem 1 of Hansen (2014), we have
\[ C^*(\mathbf{w}^*) = \sum_{m=1}^{M-1} \left\{ w_m^2 (b_m - b_{m+1}) - 2\sigma^2 w_m^2 k_{m+1} \right\} \]
\[ = \sum_{j=1}^{J-1} \sum_{\ell=m_j}^{m_{j+1}-1} \left\{ w_{\ell}^2 (b_{\ell+1} - b_\ell) - 2\sigma^2 w_{\ell}^2 k_{\ell+1} \right\} \]
\[ + \sum_{\ell=m_j}^{M-1} \left\{ w_{\ell}^2 (b_{\ell+1} - b_\ell) - 2\sigma^2 w_{\ell}^2 k_{\ell+1} \right\} \]
\[ = \sum_{j=1}^{J-1} \left\{ w_m^2 (b_{m_j+1} - b_m) - 2\sigma^2 w_m^2 (v_{m_j+1} - v_m) \right\} \]
\[ + (b_m - b_m) - 2\sigma^2 (v_m - v_m), \]
which is minimized by
\[ \hat{w}_m = \frac{\hat{\sigma}^2 (v_{m_j+1} - v_m)}{b_{m_j+1} - b_m}, \quad j = 1, \ldots, J - 1, \] (A.7)
when \( w_m \in \mathcal{X}^* \) (see (A.1) for the definition of \( \mathcal{X}^* \)).

Note that the set \( \{m_1, \ldots, m_J\} \) which contains indexes of candidate models with positive weights is random and depends on \( \hat{\beta}_{\text{OLS}} \) and \( \hat{\sigma}^2 \). When \( \hat{\beta}_{\text{OLS}} \) and \( \hat{\sigma}^2 \) vary, the set \( \{m_1, \ldots, m_J\} \) can also vary, but it is a piecewise constant function of \( \hat{\beta}_{\text{OLS}} \) and \( \hat{\sigma}^2 \) and is almost differentiable in the sense of Stein (1981) except for a finite number of points. Hence, in the following proof, when taking derivatives with respect to \( \hat{\beta}_{\text{OLS}} \) and \( \hat{\sigma}^2 \), we take \( \{m_1, \ldots, m_J\} \) to be a constant set.

Since model \( m_j \) is nested within model \( m_{j+1} \), it is easily to obtain the following results
\[ (\hat{\beta}_{m_{j+1}} - \hat{\beta}_{m_j})^T (\hat{\beta}_{M} - \hat{\beta}_{m_j}) = 0 \] (A.8)
and
\[ b_{m_{j+1}} - b_m = \left\| \hat{\beta}_{m_{j+1}} \right\|^2 - \left\| \hat{\beta}_{m_j} \right\|^2 = \left\| \hat{\beta}_{m_{j+1}} - \hat{\beta}_{m_j} \right\|^2. \] (A.9)

It follows from (2.3), (2.5)–(A.7), (A.8)–(A.9), Stein Lemma (Stein, 1981), and the independence between \( \hat{\beta}_{\text{OLS}} \) and \( \hat{\sigma}^2 \) that
\[ E \left\| \hat{\beta}(\mathbf{w}) - \beta \right\|^2 \]
\[ = E \left\| \hat{\beta}_{\text{OLS}} - \beta - \sum_{m=1}^{M-1} \hat{w}_m (\hat{\beta}_{m+1} - \hat{\beta}_m) \right\|^2 \]
\[ = E \left\| \hat{\beta}_{\text{OLS}} - \beta - \sum_{j=1}^{J-1} \sum_{\ell=m_j}^{m_{j+1}-1} \hat{w}_\ell (\hat{\beta}_{\ell+1} - \hat{\beta}_\ell) \right\|^2 \]
\[ - I(m_j < M) \sum_{\ell=m_j}^{M-1} (\hat{\beta}_{\ell+1} - \hat{\beta}_\ell)^2 \]
\[ = E \left\| \hat{\beta}_{\text{OLS}} - \beta - \sum_{j=1}^{J-1} \left\| \hat{\beta}_{m_j+1} - \hat{\beta}_m \right\|^2 \]
\[ - I(m_j < M) (\hat{\beta}_{m_j} - \hat{\beta}_m)^2 \]
\[ = E \left\| \hat{\beta}_{\text{OLS}} - \beta - \sum_{j=1}^{J-1} \left\| \hat{\beta}_{m_j+1} - \hat{\beta}_m \right\|^2 \]
\[ - I(m_j < M) (\hat{\beta}_{m_j} - \hat{\beta}_m)^2 \]
\[ = E \left\| \hat{\beta}_{\text{OLS}} - \beta \right\|^2 + E \sum_{j=1}^{J-1} \left\| \hat{\beta}_{m_j+1} - \hat{\beta}_m \right\|^2 \]
\[ + E \left[ I(m_j < M) \left\| \hat{\beta}_{M} - \hat{\beta}_{m_j} \right\|^2 \right] \]
\[ - 2E \left[ I(m_j < M) (\hat{\beta}_{m_j} - \hat{\beta}_m)^T \hat{\beta}_{\text{OLS}} - \beta \right] \]
\[ - 2E \left[ \sum_{j=1}^{J-1} \left\| \hat{\beta}_{m_j+1} - \hat{\beta}_m \right\|^2 (\hat{\beta}_{m_j+1} - \hat{\beta}_m)^T \right] \]
\[ \times \hat{\beta}_{\text{OLS}} - \beta \]
\[ + 2 \sum_{j=1}^{J-1} \left\| \hat{\beta}_{m_j+1} - \hat{\beta}_m \right\|^2 (\hat{\beta}_{m_j+1} - \hat{\beta}_m)^T \]
\[ \times (\hat{\beta}_M - \hat{\beta}_m)^T \]
\[ + E \left[ I(m_j < M) \left\| \hat{\beta}_{M} - \hat{\beta}_{m_j} \right\|^2 \right] \]
\[ - 2\sigma^2 E \left[ I(m_j < M) \text{trace } (A_{M_j} - A_m)^T X^T X (X^T X)^{-1} \right] \]
\[ - 2E \left[ \sigma^2 E \left[ \sum_{j=1}^{J-1} \left\| \hat{\beta}_{m_j+1} - \hat{\beta}_m \right\|^2 \left( \hat{\beta}_{m_j+1} - \hat{\beta}_m \right)^T \right] \right] \]
\[ \times (\hat{\beta}_{m_j+1} - \hat{\beta}_m)^T \hat{\beta}_{\text{OLS}} - \beta \]
\[ + 0 \]
\[ \sigma^2 + E \left\{ \sum_{j=1}^{J-1} (v_{mj+1} - v_{mj})^2 \right\} \\
+ 4 E \left( \sum_{j=1}^{J-1} (v_{mj+1} - v_{mj})^2 \right) \times \text{trace} \left( \left( A_{mj+1} - A_{mj} \right) X'X(X'X)^{-1} \right) \\
+ 4 \sigma^2 \left\{ \sum_{j=1}^{J-1} (v_{mj+1} - v_{mj})^2 \right\} \\
+ 4 \sigma^2 \left\{ \sum_{j=1}^{J-1} (v_{mj+1} - v_{mj})^2 \right\} \times \text{trace} \left( \left( A_{mj+1} - A_{mj} \right) X'X(X'X)^{-1} \right) \\
= \sigma^2 + E \left\{ \sum_{j=1}^{J-1} (v_{mj+1} - v_{mj})^2 \right\} \\
+ E \left[ I(m_j < M) \left\{ \left\| \hat{\beta}_{OLS} - \hat{\beta}_{OLSj} \right\|^2 - 2 \sigma^2 (v_m - v_{mj}) \right\} \right] \\
- 2 \sigma^2 \left\{ \sum_{j=1}^{J-1} (v_{mj+1} - v_{mj})^2 \right\} \\
+ 4 \sigma^2 \left\{ \sum_{j=1}^{J-1} (v_{mj+1} - v_{mj})^2 \right\} \\
+ 4 \sigma^2 \left\{ \sum_{j=1}^{J-1} (v_{mj+1} - v_{mj})^2 \right\} \times \text{trace} \left( \left( A_{mj+1} - A_{mj} \right) X'X(X'X)^{-1} \right) \\
= \sigma^2 + E \left\{ \sum_{j=1}^{J-1} (v_{mj+1} - v_{mj})^2 \right\} \\
+ E \left[ I(m_j < M) \left\{ \left\| \hat{\beta}_{OLS} - \hat{\beta}_{OLSj} \right\|^2 - 2 \sigma^2 (v_m - v_{mj}) \right\} \right] \\
- 2 \sigma^2 \left\{ \sum_{j=1}^{J-1} (v_{mj+1} - v_{mj})^2 \right\} \\
+ 4 \sigma^2 \left\{ \sum_{j=1}^{J-1} (v_{mj+1} - v_{mj})^2 \right\} \\
+ 4 \sigma^2 \left\{ \sum_{j=1}^{J-1} (v_{mj+1} - v_{mj})^2 \right\} \times \text{trace} \left( \left( A_{mj+1} - A_{mj} \right) X'X(X'X)^{-1} \right) \\
\]
When \( m_J < M \), \( I(m_J < M)(v_M - v_{m_J}) \) is larger than zero, which, along with (A.17), implies the result of Corollary 1.

References
